Magnetization plateau in the spin ladder with the four-spin exchange

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The magnetization process of the S=1/2 antiferromagnetic spin ladder with the four-spin cyclic exchange interaction at $T = 0$ is studied by the exact diagonalization of finite clusters and size scaling analyses. It is found that a magnetization plateau appears at half the saturation value if the ratio of the four- and two-spin exchange coupling constants J_4 is larger than the critical value J_{4c} =0.05±0.04. The phase transition with respect to J_4 at J_{4c} is revealed to be the Kosterlitz-Thouless-type.

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The field-induced spin gap is one of recent interesting topics on the one-dimensional (1D) quantum spin systems. The gap can be detected as a plateau of the magnetization curve at low temperatures. The appearance of such a magnetization plateau was theoretically predicted in several systems; the anisotropic $S = \frac{3}{2}$ antiferromagnetic chain [1,2], the $S = 1$ bond-alternating chain [3], the $S = \frac{1}{2}$ ferromagnetic-ferromagnetic-antiferromagnetic chain [4], the frustrated bond-alternating chain [5], the three-leg ladder [6], and the frustrated two-leg ladder [7]. In particular the two-leg ladder attracts a great interest in the context of the superconductivity in a doped system [8]. The standard $S = \frac{1}{2}$ uniform antiferromagnetic spin ladder has the spin gap of the lowest excitation from the nonmagnetic ground state (GS), which leads to a plateau at the magnetization $m = 0$ [9–11]. A strong coupling approach [7] indicated an additional plateau at half the saturated magnetization due to the next-nearest antiferromagnetic coupling which yields the frustration. In this paper we show another possibility of the magnetization plateau in the two-leg spin ladder, which is induced by a four-spin exchange interaction.

A four-spin exchange interaction described by a product of two-spin exchanges in a spin ladder was investigated by a field theoretical approach [12]. It indicated the possibility of a different type of massive phase from the Haldane phase [13] in the nonmagnetic GS, but the state in a strong magnetic field was not discussed. On the other hand a mean field analysis [14] suggested that the $S = \frac{1}{2}$ triangular lattice antiferromagnet would have a magnetization plateau at half the saturated magnetization, if there exists a four-spin cyclic exchange interaction. It was verified by the exact diagonalization [15]. The recent experiments revealed that such multiple-spin exchange interactions are realized in the two-dimensional (2D) solid ^{3}He [16,17] and the 2D Wigner solid of electrons formed in a Si inversion layer [18], as well as the bcc ^{3}He [19]. In order to test the possibility of a similar magnetization plateau in 1D quantum spin systems, we consider the $S = \frac{1}{2}$ uniform antiferromagnetic spin ladder with the four-spin cyclic exchange at every plaquette. The magnetization process of the system is described by the Hamiltonian

$$\begin{aligned}
\mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_Z, \\
\mathcal{H}_0 &= \sum_j (\mathbf{S}_{1,j}\cdot\mathbf{S}_{1,j+1} + \mathbf{S}_{2,j}\cdot\mathbf{S}_{2,j+1} + \mathbf{S}_{1,j}\cdot\mathbf{S}_{2,j}) \\
&\quad + J_4 \sum_j (P_{4,j} + P_{4,j}^{-1}), \\
\mathcal{H}_Z &= -H \sum_j (S_{1,j}^z + S_{2,j}^z),
\end{aligned} \tag{1}$$

where $P_{4,j}$ is the cyclic permutation operator which exchanges the four spins around the j-th plaquette as $\mathbf{S}_{1,j} \rightarrow \mathbf{S}_{1,j+1} \rightarrow \mathbf{S}_{2,j+1} \rightarrow \mathbf{S}_{2,j} \rightarrow \mathbf{S}_{1,j}$, J_4 is the strength of the four-spin exchange and H is the applied magnetic field normalized by the two-spin exchange coupling constant. We assume J_4 is positive, as it is in the solid ^{3}He. This system subjected to the periodic boundary condition is studied by the exact diagonalization of the finite clusters and the size scaling of the low-lying energy spectra. For $L \times 2$-spin systems, the lowest energy of $\mathcal{H}_0$ in the subspace where $\sum_j (S_{1,j}^z + S_{2,j}^z) = M$ is denoted as $E(L, M)$. Using Lanczos' algorithm, we calculated $E(L, M)$ $(M = 0, 1, 2, \cdots, L)$ for even-L systems up to $L = 16$. The macroscopic magnetization is defined as $m = \frac{M}{L}$.

The nonmagnetic GS of the system (1) with $J_4 = 0$ is in a massive phase equivalent to the Haldane phase of the $S = 1$ antiferromagnetic chain and the low-lying excitation has a finite energy gap for $m = 0$. On the other hand, the magnetic GS is always gapless [20,21] except for the saturation. Thus the magnetization curve has a plateau at $m = 0$, while no other plateau appear up to $m = 1$, as far as $J_4 = 0$. The four-spin exchange, however, is expected to induce a plateau at $m = \frac{1}{2}$, because the interaction stabilizes the '*uud*' state, mentioned in Ref. [14], of the four spins around every plaquette within a mean field argument. We concentrate on the plateau at $m = \frac{1}{2}$, rather than the nonmagnetic GS.

The magnetic excitation gap giving $\delta M = \pm 1$ of the $L \times 2$-spin systems described by the total Hamiltonian $\mathcal{H}$ is given by

$$\Delta_{\pm} \equiv E(L, M \pm 1) - E(L, M) \mp H. \quad (2)$$

For the gapless system in the thermodynamic limit, the conformal field theory [22] (CFT) predicts the asymptotic form of the size dependence of the gap as $\Delta_{\pm} \sim O(1/L)$ with fixed $m = M/L$. When H_+ and H_- are defined as

$$\begin{aligned} E(L, M+1) - E(L, M) &\to H_+ \quad (L \to \infty), \\ E(L, M) - E(L, M-1) &\to H_- \quad (L \to \infty), \end{aligned} \quad (3)$$

H_+ and H_- has the same value and it gives the magnetic field H for the magnetization m in the thermodynamic limit. In contrast to the gapless case, if the system has a finite gap even in the infinite length limit, Δ_+ and Δ_- are still finite for $L \to \infty$. It leads to the difference between H_+ and H_- and a plateau appears for $H_- < H < H_+$ at $m = M/L$ in the magnetization curve at $T = 0$.

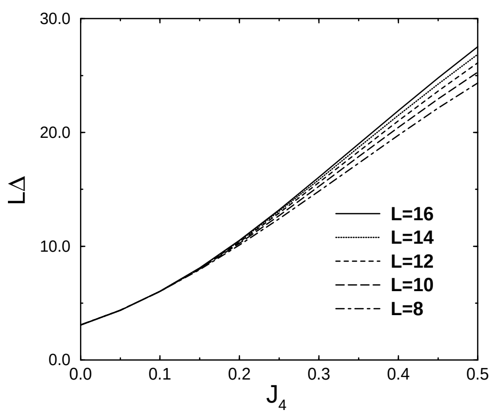

FIG. 1. Scaled gap $L\Delta$ versus the strength of the four-spin exchange interaction J_4.

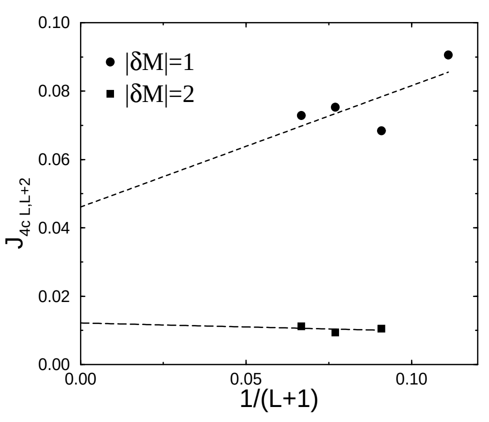

FIG. 2. L-dependent fixed point $J_{4cL,L+2}$ of the gap for $\delta M = \pm 1$ (circles) and $\delta M = \pm 2$ (squares) are plotted versus $1/(L+1)$ to determine J_{4c} in the thermodynamic limit. The estimated value is $J_{4c} = 0.05 \pm 0.01$ for $\delta M = \pm 1$, which does not well agree with the result for $\delta M = \pm 2$ $J_{4c} = 0.01 \pm 0.01$. Thus we conclude $J_{4c} = 0.05 \pm 0.04$.

The sum $\Delta \equiv \Delta_+ + \Delta_-$ is a good order parameter to investigate the plateau-nonplateau transition with the finite-size scaling [2], because Δ corresponds to the length of the plateau in the magnetization curve in the thermodynamic limit. The scaled gap $L\Delta$ of finite systems ($L = 8 \sim 16$) at $m = 1/2$ is plotted versus J_4 in Fig. 1. For $J_4 > 0.2$ the scaled gap obviously increases with increasing L, which means that a finite gap exists in the thermodynamic limit. For small J_4 around the region $0 < J_4 < 0.1$, the scaled gap looks almost independent of L. It implies that the system is gapless at a finite region of the parameter J_4, which is reminiscent of the Kosterlitz-Thouless (KT) transition [23]. According to our precise analysis, the $L\Delta$ curves for L, and $L+2$ have an intersection in the region $0 < J_4 < 0.1$ for each L. Thus the critical point J_{4c} can be estimated by the phenomenological renormalization group (PRG) equation [24]

$$(L+2)\Delta_{L+2}(J_4') = L\Delta_L(J_4). \quad (4)$$

We define $J_{4cL,L+2}$ as the L-dependent fixed point of (4) and it is extrapolated to the thermodynamic limit. $J_{4cL,L+2}$ is plotted versus $1/(L+1)$ as solid circles in Fig. 2. Although the convergence of $J_{4cL,L+2}$ with increasing L is not good, the least square fitting of the form $J_{4cL,L+2} \sim J_{4c} + A/(L+1)$ gives the extrapolated result $J_{4c} = 0.05 \pm 0.01$ as the dashed line in Fig. 2. To test the precision of the value, we did the same analysis using the gap for $\delta M = \pm 2$ instead of $\Delta_{\pm}$ as solid squares shown in Fig. 2 where the fixed point can be obtained only for $L \geq 10$. It gave $J_{4c} = 0.01 \pm 0.01$ which is not well coincide with the above result, which implies that the available system size is not enough to determine J_{4c} with the fitting of $1/(L+1)$. (Such a difficulty of the precise decision of the critical point by PRG is sometimes due to the logarithmic size correction in the case of the KT transition. [25]) Assuming that the system is gapless for $J_4 = 0$, we conclude $J_{4c} = 0.05 \pm 0.04$ within the present analysis.

We present the GS magnetization curve in the thermodynamic limit for J_4=0 and 0.1. In the latter case the magnetization plateau should appear at $m = \frac{1}{2}$ in contrast to the former, as discussed above. Note that the four-spin exchange interaction reduces the spin gap just above the nonmagnetic GS. According to our present analysis based on PRG, the gap for $m = 0$ vanishes at a critical value $\tilde{J}_{4c}$, which should be distinguished from J_{4c} for $m = \frac{1}{2}$, and the nonmagnetic GS will belong to a different massive phase from the Haldane phase for $J_4 > \tilde{J}_{4c}$. The critical value $\tilde{J}_{4c}$, however, is obviously larger than 0.1. Thus even in the case of $J_4 = 0.1$ the spin gap due to the Haldane mechanism still exists for $m = 0$ and we can use the same method to give the magnetization curve as used for the $S = 1$ antiferromagnetic chain [28].

For $J_4 = 0.1$ the left hand sides of the form (3) calculated for $m = 0, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $\frac{3}{4}$ are plotted versus $1/L$

in Fig. 3. It shows $H_+ = H_-$ except for $m = 0$ and $\frac{1}{2}$. Thus we take the mean value of the two for the magnetic field H for each m.

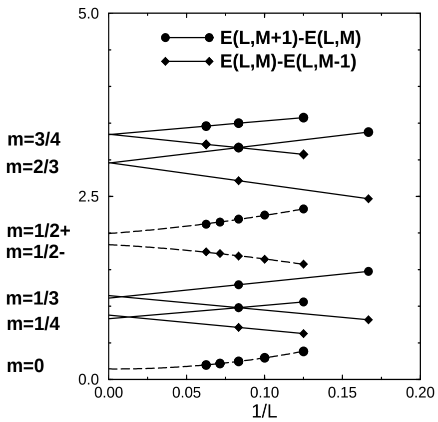

FIG. 3. $E(L,M+1)-E(L,M)$ and $E(L,M)-E(L,M-1)$ plotted versus $1/L$ with fixed m for $J_4 = 0.1$. The dashed curves are guides to the eye. The extrapolated points for $m = 0$, $m = 1/2-$ and $m = 1/2+$ correspond to the results of the Shanks' transformation $H_{c1} = 0.15 \pm 0.03$, $H_- = 1.84 \pm 0.06$ and $H_+ = 1.99 \pm 0.09$, respectively.

Since the nonmagnetic GS is massive for J_4 =0 and 0.1, the size correction of H_+ decays faster than $\frac{1}{L}$ as shown in Fig. 3. Thus we use the Shanks' transformation [29] $P'_n = (P_{n-1}P_{n+1} - P_n^2)/(P_{n-1} + P_{n+1} - 2P_n)$ twice for the sequence $E(L,1) - E(L,0)$ for L =6,8,10,12 and 14, and obtain $H_{c1} = 0.503 \pm 0.003$ and 0.15 ± 0.03 for J_4=0 and 0.1, respectively. The saturation field H_{c2} is given by the L-independent quantity $E(L,L) - E(L,L-1)$.

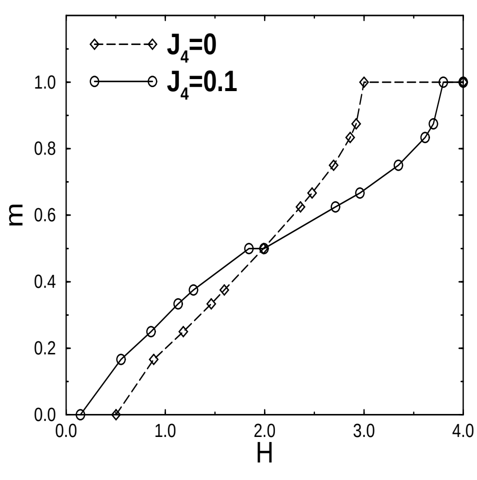

FIG. 4. GS magnetization curves in the thermodynamic limit for $J_4 = 0$ and 0.1. The latter has the magnetization plateau at half the saturation value.

In the case of $J_4 = 0.1$, for $m = 1/2$ H_+ and H_- are obviously different and the size correction decays faster than $1/L$, as shown in Fig. 3, which is consistent with a finite gap. Then we estimate H_+ and H_- by the Shanks' transformation and get $H_+ = 1.99 \pm 0.09$ and $H_- = 1.84 \pm 0.06$. For $J_4 = 0$ H_+ and H_- correspond even at $m = 1/2$. We present the results for J_4 =0 and 0.1 in Fig. 4, where we also used the values of H for m =1/6, 3/8, 5/8, 5/6 and 7/8 which are estimated by the same method as mentioned above. The curve has a plateau at $m = 1/2$ ($H_- < H < H_+$) for J_4 =0.1.

Our present PRG analysis shows that the the gap does not behaves as $\Delta \sim (J_4 - J_{4c})^\nu$. If we define the the size-dependent exponent ν_L, it diverges as L increases. Instead, if the gap behaves like

$$\Delta \sim \exp\big(-\frac{a}{(J_4 - J_{4c})^\sigma}\big), \tag{5}$$

as in the case of universality class of KT transitions ($\sigma = \frac{1}{2}$), the Roomany-Wyld approximation for the Callen-Symanzik β-function [26], which is defined as

$$\beta_{L,L+2}(J_4) = \frac{1 + \log\big(\frac{\Delta_{L+2}(J_4)}{\Delta_L(J_4)}\big)/\log\big(\frac{L+2}{L}\big)}{\big[\frac{\Delta'_L(J_4)\Delta'_{L+2}(J_4)}{\Delta_L(J_4)\Delta_{L+2}(J_4)}\big]^{\frac{1}{2}}}, \tag{6}$$

should have the form

$$\beta_{L,L+2}(J_4) \sim (J_4 - J_{4cL,L+2})^{1+\sigma}. \tag{7}$$

Fitting the form (7) to the calculated function (6) for each L, σ is estimated as follows: $\sigma_{10,12} = 0.38 \pm 0.10$, $\sigma_{12,14} = 0.43 \pm 0.10$ and $\sigma_{14,16} = 0.49 \pm 0.10$. The results are consistent with $\sigma = \frac{1}{2}$. Thus we conclude the critical behavior near J_{4c} for $m = \frac{1}{2}$ is characterized by the universality class of the KT transition.

Furthermore we estimate the central charge c of CFT, using the asymptotic form of the GS energy per site

$$\frac{1}{L}E(L,M) \sim \epsilon(m) - \frac{\pi}{6}cv_s\frac{1}{L^2} \qquad (L \to \infty), \tag{8}$$

where v_s is the sound velocity which is the gradient of the dispersion curve at the origin. The result shown in Fig. 5 suggests $c = 1$ with only a few percent errors for $m = \frac{1}{2}$ and $0 \le J_4 \le 0.1$. It also supports the KT transition.

The critical exponent η, associated with the spin correlation function in the leg direction like $\langle S_0^+ S_r^- \rangle \sim (-1)^r r^{-\eta}$, can be estimated by the form of the gap $\Delta_\pm \sim \pi v_s \eta / L \quad (L \to \infty)$ [2]. The estimated η, shown in Fig. 5, seems close to $\frac{1}{2}$ around the critical point J_{4c}, rather than $\frac{1}{4}$ which is expected for the KT transition. We think there is a possible jump from $\eta = \frac{1}{4}$ to $\eta = \frac{1}{2}$ at J_{4c} because the elementary excitations is expected to behave like the free Fermion systems ($\eta = \frac{1}{2}$) at the edge of the plateau for $J_4 > J_{4c}$ [5]. The present small cluster analysis could not detect such a discontinuity. Another exponent η^z defined as $\langle S_0^z S_r^z \rangle \sim \cos(2k_F r) r^{-\eta^z}$ can also be estimated from the L-dependence of the soft mode

gap with the momentum $2k_F = 2\pi m$ [21]. We checked the validity of the relation $\eta\eta^z = 1$ around J_{4c} which is consistent with the Luttinger liquid theory [27] leading to $\eta = \frac{1}{2}$ in the free Fermion case.

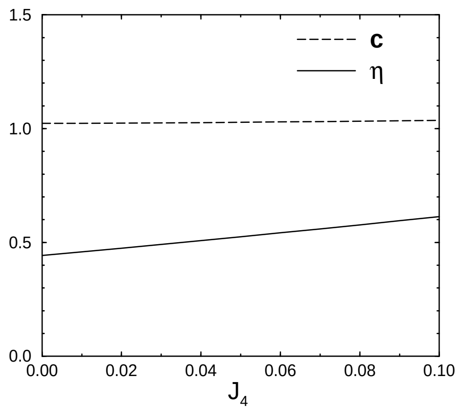

FIG. 5. Estimated central charge c and exponent η around J_{4c}. The result indicates $c = 1$ which is consistent with the KT transition. η is close to $\frac{1}{2}$ rather than $\frac{1}{4}$.

The spin gap at $m = 0$ has already been observed in several real ladder compounds, for example $Cu_2(C_2H_{12}N2)_2Cl_4$ [30,20] and $La_6Ca_8Cu_{24}O_{41}$ [31]. The magnetization plateau, however, has not been detected at any finite magnetization. We hope some new ladder materials with the field-induced spin gap will be discovered in the near future.

In summary the finite cluster calculation and size scaling study showed that the $S = \frac{1}{2}$ antiferromagnetic spin ladder with the four-spin cyclic exchange interaction at every plaquette has the magnetization plateau at $m = 1/2$ for $J_4 > J_{4c} = 0.05 \pm 0.04$ and the phase transition with respect to J_4 belongs to the KT universality class.

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